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A numerical study of a converging cylindrical shock

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A numerical proceure is introduced to solve the one-dimensional equations of gas-dynamics for a cylindrically or spherically symmetric flow. The method consists of a judicious combination of Glimm's method and operator splitting. The method is applied to the problem of a converging cylindrical shock.

1. Introduction

The one-dimesional equations for an inviscid, non-heat-conducting, radially symmetric flow can be written in the form

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_r = -\mathbf{W}(\mathbf{U}), \tag{1}$$

where

$$\mathbf{U} = \begin{pmatrix} \rho \\ m \\ e \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} m \\ m^2/\rho + p \\ m(e+p)/\rho \end{pmatrix}, \quad \mathbf{W}(\mathbf{U}) = (\alpha - 1)\begin{pmatrix} m/r \\ m^2/\rho r \\ m(e+p)/\rho r \end{pmatrix}, \tag{2}$$

where ρ is the density, u is the velocity, $m = \rho u$ is the momentum, p is the pressure, e is the energy per unit volume, t is time, r is the space co-ordinate of symmetry, α is a constant which is equal to 2 for cylindrical symmetry and 3 for spherical symmetry, and the subscripts indicate differentiation. We may write

$$e = p/(\gamma - 1) + \tfrac{1}{2}\rho u^2, \tag{3}$$

where γ is the ratio of specific heats (a constant greater than 1).

There are two major problems involved in solving the system (1) directly. The first is its singular nature near the axis ($r = 0$), i.e. there are singular terms proportional to $1/r$. The second problem is that the momentum equation [the second component equation of (1)] cannot be put in conservation form.

These problems cause major difficulties near the axis. These are usually overcome by some *ad hoc* method such as extrapolation (Payne 1956). Another approach has been to treat this as a problem in Cartesian co-ordinates in two space dimensions (Lapidus 1971).

In the method described below both of these problems have been completely eliminated. Thus there is no need to resort to any trickery in order to solve the system (1).

2. Outline of the method

The first step in the problem is to use the method known as operator splitting to remove the inhomogeneous terms $-\mathbf{W}(\mathbf{U})$ from the system (1). Thus we solve the system

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_r = 0, \tag{4}$$

which represents the one-dimensional equations of gasdynamics in Cartesian co-ordinates.

The method used to solve system (4) is the random-choice method introduced by Glimm (1965) and developed for hydrodynamics by Chorin (1976). Details of this method will be given in the next section, for completeness.

Once system (4) has been solved, the system of ordinary differential equations

$$\mathbf{U}_t = -\mathbf{W}(\mathbf{U}) \tag{5}$$

is solved, the solution of system (4) being used to determine the inhomogeneous term $-\mathbf{W}$ in (5). There are several reasons for this approach, which will be discussed in later sections.

3. Glimm's method

Consider the nonlinear system of equations (4). Divide time into intervals of length Δt and let Δr be the spatial increment. The solution is to be evaluated at times $n\Delta t$, where n is a non-negative integer, at the spatial points $i\Delta r$, where $i = 0, \pm 1, \pm 2, \ldots$, and at times $(n+\frac{1}{2})\Delta t$ at $(i+\frac{1}{2})\Delta r$.

The method is a two-step method. Let $\tilde{\mathbf{u}}_i^n$ approximate $\mathbf{U}(i\Delta r, n\Delta t)$ and $\tilde{\mathbf{u}}_{i+\frac{1}{2}}^{n+\frac{1}{2}}$ approximate $\mathbf{U}((i+\frac{1}{2})\Delta r, (n+\frac{1}{2})\Delta t)$ in (4). To find the solution $\tilde{\mathbf{u}}_{i+\frac{1}{2}}^{n+\frac{1}{2}}$, consider the system (4) along with the piecewise-constant initial data

$$\mathbf{U}(r, n\Delta t) = \begin{cases} \tilde{\mathbf{u}}_{i+1}^n, & r > (i+\frac{1}{2})\Delta r, \\ \tilde{\mathbf{u}}_i^n, & r < (i+\frac{1}{2})\Delta r. \end{cases} \tag{6}$$

This gives a sequence of Riemann problems. If $\Delta t < \Delta r/2(|u|+c)$, where c is the local sound speed, the waves generated by the different Riemann problems will not interact. Hence the solution $\mathbf{v}(r, t)$ to the Riemann problem can be combined into a single exact solution; see figure 1. Let ξ_n be an equidistributed random variable which is given by the Lebesgue measure on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Define

$$\tilde{\mathbf{u}}_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \mathbf{v}((i+\xi_n)\Delta r, (n+\tfrac{1}{2})\Delta t); \tag{7}$$

see figure 2.

At each time step, the solution is approximated by a piecewise-constant function. The solution is then advanced in time exactly and the new values are sampled. The method depends on solving the Riemann problem exactly and inexpensively.

Chorin (1976; see also Sod 1976, 1977) modified an iterative method due to Godunov (1959), which will now be described. Consider the system (4) with the initial data

$$\mathbf{U}(r, 0) = \begin{cases} S_l = (\rho_l, u_l, p_l), & r < 0, \\ S_r = (\rho_r, u_r, p_r), & r \geqslant 0. \end{cases} \tag{8}$$

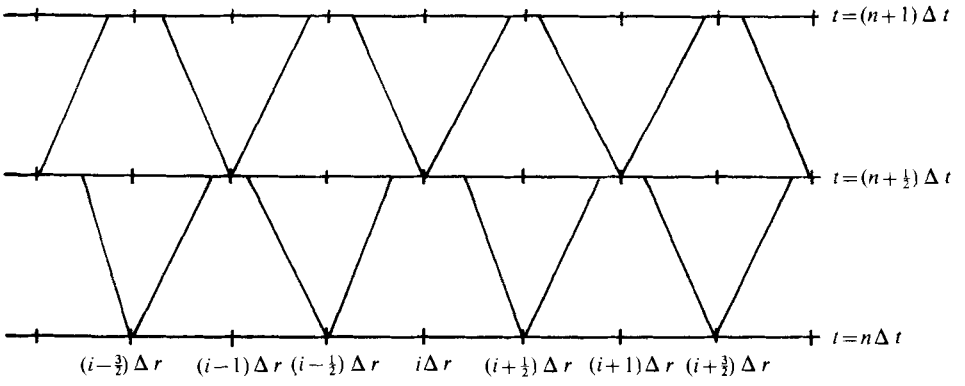


FIGURE 1. Sequence of Riemann problems on grid.

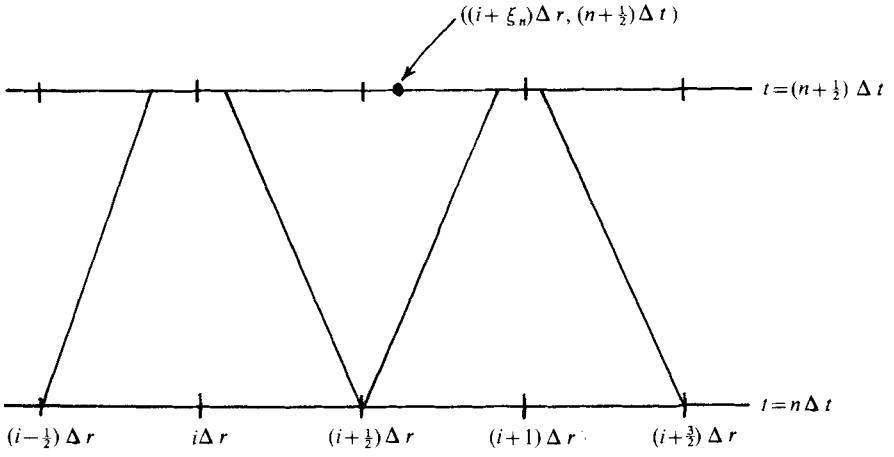


FIGURE 2. Sampling procedure for Glimm's scheme.

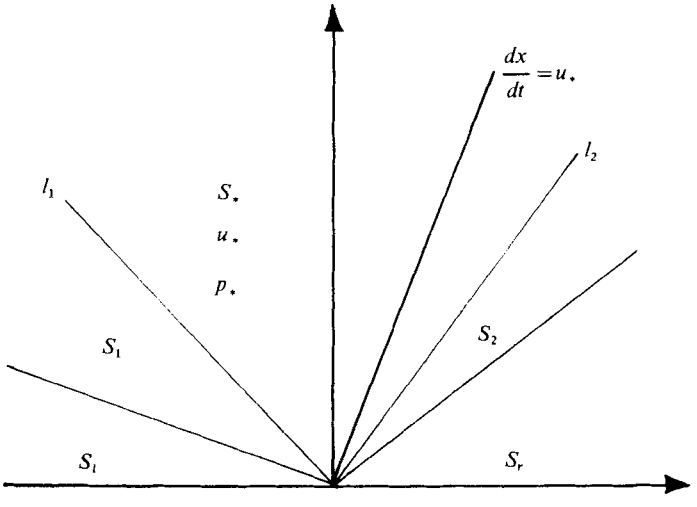


FIGURE 3. Solution of Riemann problem.

The solution at later times looks like figure 3, where S_1 and S_2 are either a shock or a centred rarefaction wave. The region S_* is a steady state. The lines l_1 and l_2 separate the states. The contact surface $dr/dt = u_*$ separates the region into two parts with possibly different values of ρ_*, but equal values of u_* and p_*.

Using this iterative method we first evaluate p_* in the state S_*. Define the quantity

$$M_l = (p_l - p_*)/(u_l - u_*). \tag{9}$$

If the left wave is a shock, using the jump condition $U_l[\rho] = [\rho u]$ we obtain

$$M_l = \rho_l(u_l - U_l) = \rho_*(u_* - U_l), \tag{10}$$

where U_l is the velocity of the left shock and ρ_* is the density in the portion of S_* adjoining the left shock. Similarly, define the quantity

$$M_r = (p_r - p_*)/(u_r - u_*). \tag{11}$$

If the right wave is a shock, using the jump condition $U_r[\rho] = [\rho u]$ we obtain

$$M_r = -\rho_r(u_r - U_r) = -\rho_*(u_* - U_r), \tag{12}$$

where U_r is the velocity of the right shock and ρ_* is the density in the portion of S_* adjoining the right shock.

In either case [(9) or (10) for M_l and (11) or (12) for M_r] we obtain

$$M_l = (\rho_l p_l)^{\frac{1}{2}} \phi(p_*/p_l), \tag{13}$$

$$M_r = (\rho_r p_r)^{\frac{1}{2}} \phi(p_*/p_r), \tag{14}$$

where

$$\phi(x) = \begin{cases} \left(\dfrac{\gamma+1}{2}x + \dfrac{\gamma-1}{2}\right)^{\frac{1}{2}}, & x \geqslant 1, \\ \dfrac{\gamma-1}{2\gamma^{\frac{1}{2}}} \dfrac{1-x}{1-x^{\gamma-\frac{1}{2}\gamma}}, & x \leqslant 1. \end{cases} \tag{15}$$

Upon elimination of u_* from (9) and (11) we obtain

$$p_* = \frac{u_l - u_r + p_l/M_l + p_r/M_r}{1/M_l + 1/M_r}. \tag{16}$$

Equations (13), (14) and (16) are three equations in three unknowns for which there exists a real solution. After choosing a starting value p_*^0 (or M_l^0 and M_r^0), we iterate using these three equations. Here we choose $p_*^0 = \frac{1}{2}(p_l + p_r)$ (for details see Chorin 1976; or Sod 1976).

After p_*, M_l, and M_r have been determined we may obtain u_* by eliminating p_* from (9) and (10):

$$u_* = \frac{p_l - p_r + M_l u_l + M_r u_r}{M_l + M_r}. \tag{17}$$

For a discussion of the method of choosing the random numbers most efficiently, see Chorin (1976).

4. Solution of the ordinary differential equations

Once the solution $\tilde{\mathbf{u}}_i^{n+1}$ of (4) has been obtained, we have to solve the system of ordinary differential equations (5). We approximate (5) by

$$(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)/\Delta t = \mathbf{W}(\tilde{\mathbf{u}}_i^{n+1}),$$

or

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \Delta t \mathbf{W}(\tilde{\mathbf{u}}_i^{n+1}). \tag{18}$$

This approximation (18) is the basic Cauchy–Euler scheme, which is just first-order accurate. However, the Glimm scheme is at most first-order accurate so there is no reason for using a high-order method for solving the system of ordinary differential equations.

Since this system (5) is solved only at interior points and the scheme (18) does not require values at $r = 0$, the singularity at the axis is eliminated.

5. Boundary conditions

Boundary conditions need be applied only to the system (4) since the system of ordinary differential equations (5) uses only interior points. So the procedure described by Chorin (1976) readily handles the boundary condition at the axis ($r = 0$). The boundary condition is imposed at the grid point closest to $r = 0$, say $i_0 \Delta r$. A fake left state is created at $(i_0 - \frac{1}{2})\Delta r$ by setting

$$\tilde{\rho}_{i_0-\frac{1}{2}} = \tilde{\rho}_{i_0+\frac{1}{2}}, \quad \tilde{u}_{i_0-\frac{1}{2}} = -\tilde{u}_{i_0+\frac{1}{2}}, \quad \tilde{\rho}_{i_0-\frac{1}{2}} = \tilde{\rho}_{i_0+\frac{1}{2}}.$$

In this way the shock or rarefaction wave will reflect, which on the average is exact.

6. Application to a converging cylindrical shock

Initially a cylindrical diaphragm of radius r_0 separates two uniform regions of gas at rest as in a shock tube, the outer pressure and density being larger than the inner ones. After the diaphragm is ruptured ($t > 0$), a shock wave is created and travels into the low pressure region followed by a contact discontinuity. A rarefaction wave travels into the high pressure region. See figure 4.

It is known that a cylindrical shock wave in a compressible fluid increases in strength as it converges towards the axis. This can be seen experimentally in Perry & Kantrowitz (1951).

In the example given below the pressure and the density in the inner region were set equal to 1·0 and the pressure and density in the outer region were set equal to 4·0. This will produce a shock with an initial strength of 1·93, a contact discontinuity and a rarefaction wave. We took $\Delta r = 0{\cdot}01$. The time step Δt is chosen such that the Courant–Friedrichs–Lewy condition is satisfied, i.e.

$$\max\,(|u| + c)\Delta t/\Delta r \leqslant 1,$$

where c is the local sound speed.

In figure 5(*a*) the pressure distribution is displayed at time intervals of 0·05. The shock appears as a rapid variation in p which is completely sharp, i.e. the number of zones over which this variation takes place is zero. As time increases the shock

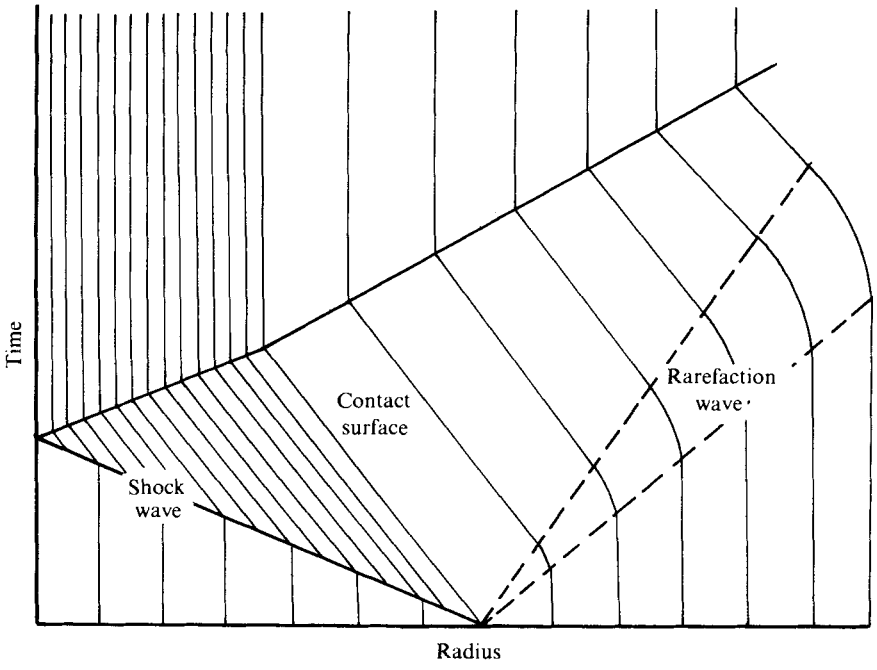


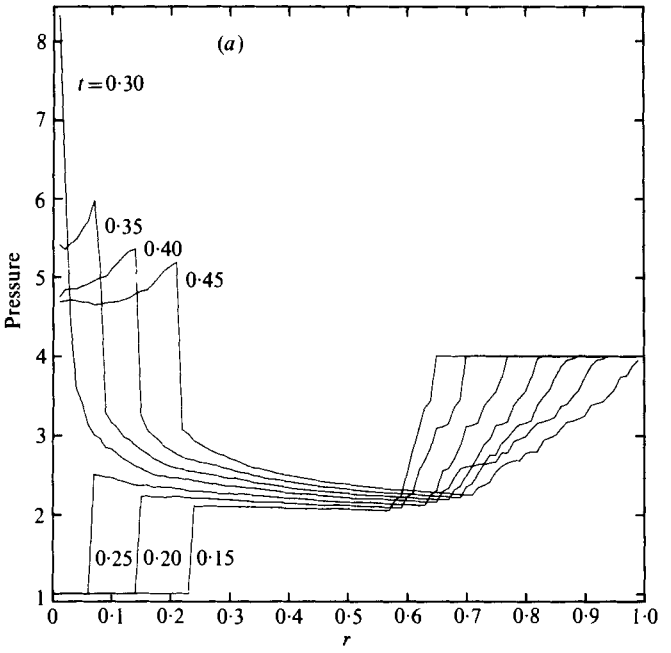
FIGURE 4. Flow pattern for converging cylindrical shock.

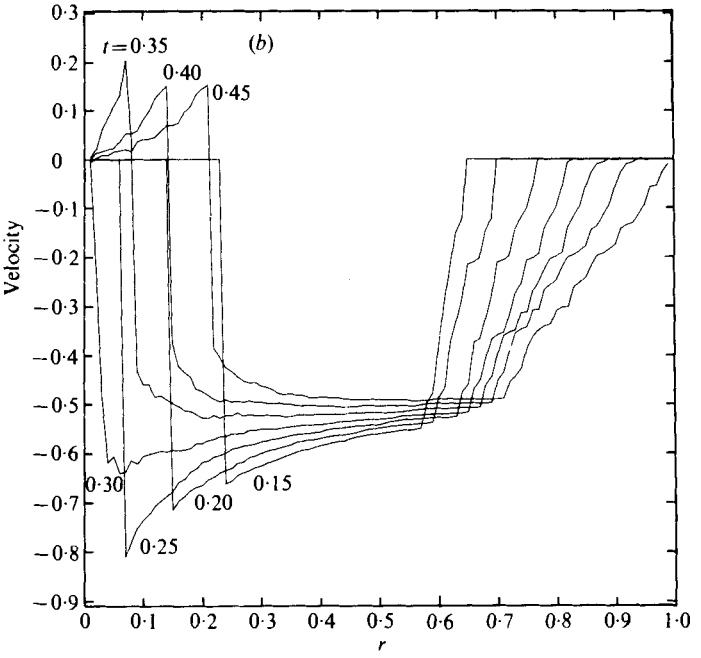
propagates towards the axis. It is observed that the strength of the shock increases with time. After the passage of the shock, the pressure behind the shock increases. When the shock arrives at the axis it is reflected, its strength rises to a large but finite value and a diverging shock appears. It is also observed that the pressure at a given point behind the reflected shock decreases with time.

In figure 5(*b*) the velocity of the gas is displayed. The behaviour is similar to that of the pressure except that the converging shock decreases the velocity from zero to a negative value. When the shock is reflected from the axis, the diverging shock has the effect of producing a small positive (outward) velocity. As in the case of the pressure profile, at a given point behind the converging shock the velocity increases with time and at a given point behind the diverging shock the velocity decreases with time.

The density and energy profiles are displayed in figures 5(*c*) and (*d*) respectively. The basic properties of the shock are similar to those of the pressure distribution, except that the rise in density across the shock is smaller owing to a temperature increase. In the density and energy profiles a contact discontinuity appears. It is a result of using Glimm's scheme that the contact discontinuity (as well as the shock wave) is completely sharp. The contact discontinuity propagates towards the axis behind the converging shock and is traversed by the reflected (outgoing) shock.

Figure 6 shows the density profile where the contact discontinuity and the reflected shock wave have crossed. For a polytropic gas with the same values of γ, higher sound speeds correspond to higher densities (Courant & Friedrichs 1948, p. 179). The interaction of a diverging shock wave and a contact discontinuity propagating towards





FIGURES 5(*a*) and (*b*). For legend see page 792.

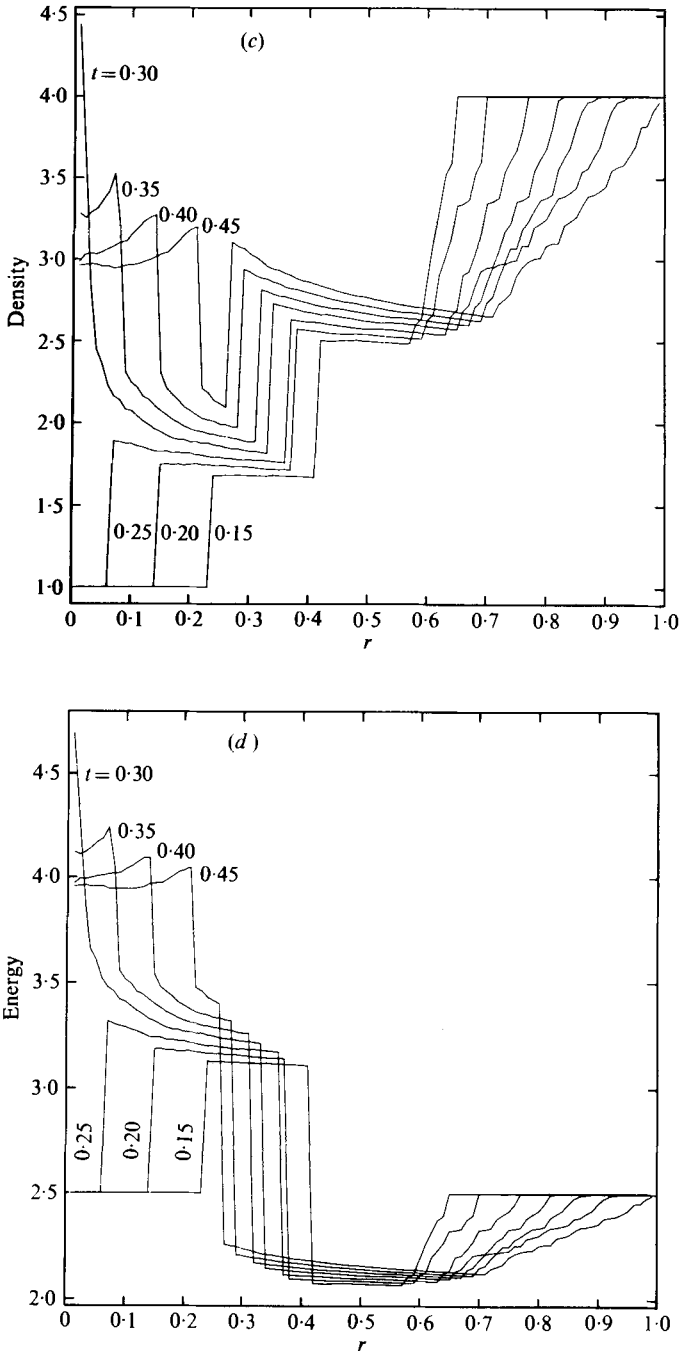


FIGURE 5. (*a*) Pressure, (*b*) velocity, (*c*) density and (*d*) energy profiles at time intervals of 0·05.

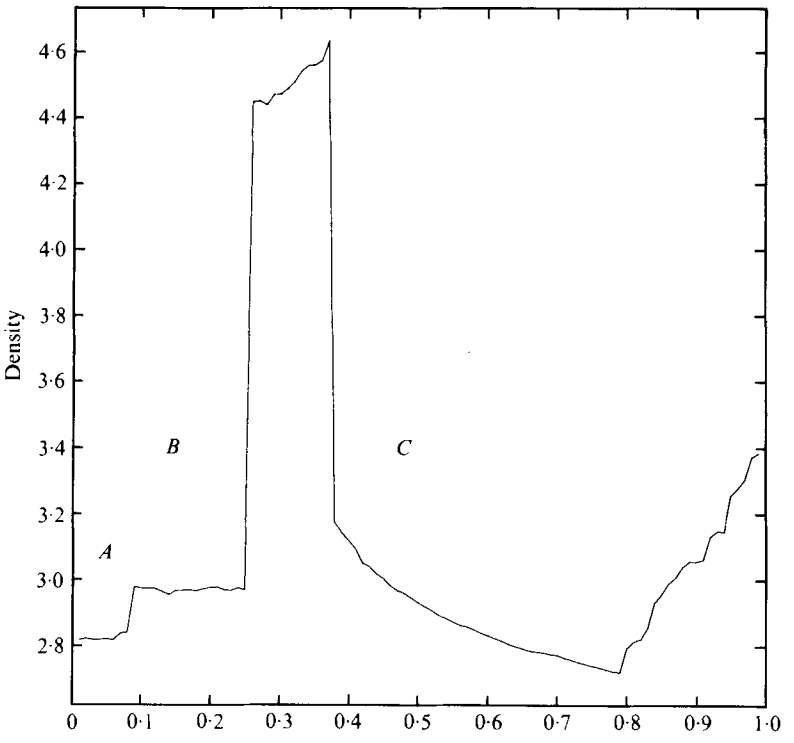


FIGURE 6. Density profile after interaction of diverging shock and contact discontinuity at time $t = 0{\cdot}6$.

the axis results in a reflected (converging) shock (represented by A), a contact discontinuity propagating towards the axis (represented by B) and a transmitted (diverging) shock (represented by C).

In general, the overall trend of the results agrees with those of Abarbanel & Goldberg (1972), Lapidus (1971) and Payne (1956). There is, however, one major difference: the time at which the shock reaches the axis. Our method is in agreement with the method of Abarbanel & Goldberg. However, for the methods of Lapidus and Payne the shock reaches the axis sooner.

It should be noted that, as a result of the randomness of Glimm's method, at a given time the position of the shock or contact discontinuity may not be exact. Yet on the average their positions are exact.

With the three other methods used in the comparison, the shock and contact discontinuity are smeared. The smearing of the shock is less dramatic. The contact discontinuity obtained by Payne's method is almost immediately smeared to such a degree that it is barely visible. However, our technique produces perfectly sharp shocks and contact discontinuities.

As discussed above, the interaction of the reflected shock and the contact discontinuity will produce a contact discontinuity, a transmitted shock and a reflected shock. The reflected shock is produced by our technique (see figure 6) but is not produced by the methods of Abarbanel & Goldberg, Lapidus and Payne.

7. Conclusions

This method reduces the problem of solving the one-dimensional equations of gasdynamics for a cylindrically or spherically symmetric flow to solving the one-dimensional equations of gasdynamics in Cartesian co-ordinates and a single system of ordinary differential equations, by using operator splitting.

The equations of gasdynamics are solved using Glimm's method, which keeps the shock waves and contact discontinuities perfectly sharp. The ordinary differential equations are solved using the Cauchy–Euler scheme at the interior points only and for one time step. Thus the singular nature of the original system near the axis is eliminated. Since the equations of gasdynamics are solved in Cartesian co-ordinates the momentum equation can be written in conservation form.

It should be noted that the roughness in the rarefaction wave is a result of the randomness of the Glimm scheme.

In all our calculations there were 100 spatial grid points, and it takes about 10·3 s on a CDC 7600 to complete 300 time steps.†

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† The computer program used to obtain the results above is available from the author.

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